The Tridiagonal Forms of Real, Nonderogatory Matrices

JOHN W. RAINEY
University of Kentucky
Lexington, Kentucky

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1. It is well known (see, e.g., [1, 6]) that any square complex matrix is similar to a tridiagonal matrix ($T = [t_{ij}]_{i,j=1}^{n}$ is tridiagonal if $t_{ij} = 0$ when $|i - j| > 1$). One can easily show that there exist real, symmetric, tridiagonal forms of Hermitian matrices; more generally there exist irreducible,* Hermitian, tridiagonal forms for matrices which have real, simple eigenvalues (see, e.g., [5]). The latter fact is a special case of the primary result of this paper:

(A) *Let M be any $n \times n$ ($n \geqslant 2$), real, nonderogatory*† *matrix. Then M is similar to a real, irreducible, tridiagonal matrix, T, where, for some integer, $k(1 \leqslant k \leqslant n - 1)$, the submatrices $[t_{ij}]_{i,j=1}^{k}$ and $[t_{ij}]_{i,j=k+1}^{n}$ are symmetric. T is not unique; k is not unique if M has at least one real eigenvalue with odd multiplicity.*

We shall verify (A) by first proving a certain decomposition theorem (Theorem 2) for real polynomials and then showing, in Section 3, that the theorem plus a lemma implies (A).

2. In the following, capital Latin letters subscripted with small letters will represent complex polynomials in the complex parameter, λ, with degrees equal to the subscripts.

The following theorem is a composite of two results which appear in [4] and the reader is referred to [4, Satz 5.4 and Satz 5.22] for their proofs.

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* $t_{j+1,j}t_{j,j+1} \neq 0$ for $j = 1, \ldots, n - 1$.
† The minimum and characteristic polynomials of M are identical.

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THEOREM 1. *Let F_j ($j \geqslant 2$) and F_{j-1} have real coefficients. The zeros of F_jF_{j-1} are real and simple and the zeros of F_{j-1} interlace those of F_j if and only if the polynomial*

$$F_j + iF_{j-1} \qquad (i^2 = -1)$$

has zeros which lie on one side of the real line.

We now want to prove

THEOREM 2. *Let P_n ($n \geqslant 2$) be real and monic. Then there exists an integer k ($1 \leqslant k \leqslant n-1$), real, monic polynomials P_k, P_{k-1}, Q_{n-k}, and Q_{n-k-1}, and a real, nonzero parameter, ρ, such that*

(i) $P_n = P_kQ_{n-k} + \rho P_{k-1}Q_{n-k-1}$;

(ii) *the zeros of P_kP_{k-1} and $Q_{n-k}Q_{n-k-1}$ are real and simple;*

(iii) *If $k \geqslant 2$ the zeros of P_{k-1} interlace those of P_k and if $n-k \geqslant 2$ the zeros of Q_{n-k-1} interlace those of Q_{n-k}.*

Proof. For $0 \leqslant m \leqslant n-1$, let $R_m (\not\equiv 0)$ be any real polynomial which does not have any real zero in common with P_n. Then H_n, defined by

$$H_n = P_n + iR_m, \tag{2.1}$$

is monic and has no real zeros. Since P_n and R_m have real coefficients note that we may write

$$P_n(\lambda) = \tfrac{1}{2}[H_n(\lambda) + \bar{H}_n(\bar{\lambda})], \tag{2.2}$$

the upper bar denoting complex conjugation.

Since H_n has no real zeros we may assume that for some k ($1 \leqslant k \leqslant n-1$)

$$H_n(\lambda) = F_k(\lambda)\bar{G}_{n-k}(\bar{\lambda}), \tag{2.3}$$

where F_k and G_{n-k} are monic and all of the zeros of either H_n or F_kG_{n-k} lie on one side of the real line. Let α and β be the negatives of the sums of the imaginary parts of the zeros of F_k and G_{n-k}, respectively, and let

$$\rho = \alpha\beta; \tag{2.4}$$

from the preceding remarks ρ must be nonzero.

Now let

$$P_k(\lambda) = \tfrac{1}{2}[F_k(\lambda) + \bar{F}_k(\bar{\lambda})], \tag{2.5}$$

$$P_{k-1}(\lambda) = (2i\alpha)^{-1}[F_k(\lambda) - \bar{F}_k(\bar{\lambda})], \tag{2.6}$$

$$Q_{n-k}(\lambda) = \tfrac{1}{2}[G_{n-k}(\lambda) + \bar{G}_{n-k}(\bar{\lambda})], \tag{2.7}$$

$$Q_{n-k-1}(\lambda) = (2i\beta)^{-1}[G_{n-k}(\lambda) - \bar{G}_{n-k}(\bar{\lambda})]. \tag{2.8}$$

P_k and Q_{n-k} are obviously real and monic; since α and β are the imaginary parts of the coefficients of λ^{k-1} in F_k and λ^{n-k-1} in G_{n-k}, respectively, P_{k-1} and Q_{n-k-1} are also real and monic.

Now from (2.5) and (2.6)

$$P_k + i\alpha P_{k-1} = F_k \tag{2.9}$$

and from (2.7) and (2.8)

$$Q_{n-k} + i\beta Q_{n-k-1} = G_{n-k}, \tag{2.10}$$

so that, from Theorem 1, the polynomials defined by (2.5)–(2.8) satisfy conditions (ii) and (iii) of our theorem; by noting (for example) that

$$\bar{F}_k(\bar{\lambda}) = P_k(\lambda) - i\alpha P_{k-1}(\lambda),$$

we can now use (2.2)–(2.4) and (2.9), (2.10) in an obvious manner to show that these polynomials also satisfy condition (i) of the theorem and thereby complete our proof.

The following theorem is simply a restriction of the Cauchy index theorem (see [2, p. 129]) to (2.1) and we omit the proof.

THEOREM 3. *As λ varies on the real axis from $-\infty$ to $+\infty$, let σ be the number of real zeros of P_n at which P_n/R_m changes from $-$ to $+$, and τ the number of real zeros of P_n at which P_n/R_m changes from $+$ to $-$. Then H_n has $\frac{1}{2}[n + (\tau - \sigma)]$ zeros in the upper half-plane.*

Since P_n and R_m have no common real zero, the quotient P_n/R_m can change sign only at those real zeros of P_n which have odd multiplicities; we could choose R_m such that the sign changes at such zeros were at our disposal. This last statement, in conjunction with Theorem 3, suggests an obvious proof of the following:

LEMMA. *Let P_n have exactly s distinct, real zeros with odd multiplicities. Then the integer k noted in Theorem 2 can assume any given value in the range* $\max\{1, \frac{1}{2}(n - s)\}$ to $\min\{n - 1, \frac{1}{2}(n + s)\}$.

3. Let P_n be the characteristic polynomial of a real nonderogatory matrix, M, and let (i) in Theorem 2 represent a decomposition of P_n which satisfies (ii) and (iii).

As is well known (from the theory of continued fractions and its connections with the determinants of tridiagonal matrices), (ii) and (iii) of Theorem 2 imply the existence of symmetric, tridiagonal matrices

$$\Delta_k = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{k-1} \\ & & b_{k-1} & a_k \end{bmatrix}, \qquad \Gamma_{k+1} = \begin{bmatrix} a_{k+1} & b_{k+1} & & \\ b_{k+1} & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix}$$

such that $P_k(\lambda)$, $Q_{n-k}(\lambda)$, $P_{k-1}(\lambda)$, and $Q_{n-k-1}(\lambda)$ are the respective characteristic polynomials of Δ_k, Γ_{k+1}, Δ_{k-1}, and Γ_{k+2}.

From an identity due to Muir [3, p. 156] the right-hand side of (i) in Theorem 2 represents the characteristic polynomial of the tridiagonal matrix

$$T = \begin{bmatrix} \Delta_k & \\ & \beta & \\ & -\alpha & \\ & & \Gamma_{k+1} \end{bmatrix}, \tag{3.1}$$

where $\alpha\beta = \rho$. Since T is nonderogatory,* T and M have the same minimum and characteristic polynomials and are therefore similar. From the proof of Theorem 2, T is obviously not unique; from the lemma, k is not fixed if M has at least one real eigenvalue with odd multiplicity. (A) is therefore a consequence of the theorem and lemma.

REFERENCES

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* Let E be the n-dimensional row vector, $[1\ 0\ 0\ \ldots\ 0]$. One can easily show that the vectors, $E, ET, \ldots, ET^{n-1}$ are linearly independent and therefore the degree of the minimum polynomial of T must be n.

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